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NEGATION AS A MODAL OPERATOR *

Abstract. Kripke-style models which besides the intuitionistic accessibility relation have a modal accessibility relation, and in which negation is treated as a modal impossibility operator, are given for propositional logics with negation weaker than Johansson's negation, as well as for Johansson's and Heyting's propositional logics and their extensions. The weakest logic captured by these models — that one in which the modal relation is as general as possible — is properly contained in Johansson's logic. Models of this type adequate for the Johansson propositional calculus are shown intertranslatable with the standard Kripke models for this calculus. Conditions which must be met by models of this type to capture various negation axioms, and some known extensions of the Johansson propositional calculus with these axioms, are also considered. It is shown how in models adequate for the Heyting propositional calculus the modal relation becomes definable in a certain sense in terms of the intuitionistic relation. Finally, some comments are made on models of this type for propositional calculi based on classical or intermediate negationless logics. *AMS Subject Classification (1980)*: 03B20 Fragments of classical logic, 03B55 Intermediate logics, 03B45 Modal logic.

§ 0. Introduction

The Johansson propositional calculus, which is obtained by weakening negation in the Heyting propositional calculus, is sound and complete with respect to Kripke models with a hereditary set of "queer" worlds Q in which the absurd holds (cf. § 5). This is the weakest logic we can capture with Kripke models with Q, and these models become inapplicable in the study of systems obtained by weakening negation still further. In this paper we shall investigate Kripke-style models which can be used not only for Johansson's and

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Heyting's propositional logics and their extensions, but also for these weak negation systems.

These models will be Kripke models with an extra R_N relation, besides the intuitionistic reflexive and transitive relation R_I, and we shall have

$$x \models A \to B \Leftrightarrow \forall y(xR_I y \Rightarrow (y \models A \Rightarrow y \models B))$$
$$x \models \neg A \quad \Leftrightarrow \forall y(xR_N y \Rightarrow y \not\models A)\,.$$

This amounts to treating negation as a modal impossibility operator added to negationless logic, which is in this case the negationless fragment of the Heyting propositional calculus. By making R_I an equivalence, or an identity relation, we easily pass from these models to models for extensions of the negationless fragment of the classical propositional calculus with various negation axioms. We can also put on R_I weaker conditions than that in order to obtain intermediate logics.

We shall first consider the weakest propositional logic captured by these models, i.e., the logic obtained when the R_N relation is as general as possible. This logic, which is properly included in Johansson's, will be called **N**. Next we shall consider models with R_N adequate for the Johansson propositional calculus. These last models will be shown inter-translatable with models which have Q. Finally, we shall consider conditions which must be put on the R_N relation to capture various negation axioms. We shall consider some known extensions of the Johansson propositional calculus with these axioms, and models with R_N adequate for these extensions. We shall also show how when we reach the Heyting propositional calculus, R_N becomes definable in a certain sense in terms of R_I. At the very end we shall briefly consider models with R_N where the R_I relation is strengthened in the sense indicated above.

To obtain models for systems with negation still weaker than negation in **N** one could try to adapt the neighbourhood semantics for modal logic (see [4], Chapter 7). However, we shall not try to do that here. Models treated in this paper correspond to the semantics for normal modal logics.

This paper applies to intuitionistic and stronger logics a technique for treating negation which was explored with relevant logics in [1]. It is also connected with [3], [6], [7], [8] and [2], where models for intuitionistic modal logics — quite similar to the models treated here — were explored in some detail. The general background of this paper is provided by [11].

§ 1. The syntax of N

The systems we shall consider will be formulated in a standard propositional language which we shall call L. In L we have denumerably many propositional variables, for which we use the schemata $p, q, r, p_1, \ldots$; the connectives of L are $\to$, $\wedge$, $\vee$ and $\neg$. We use $A, B, C, \ldots, A_1, \ldots$ as schemata for formulae of L. Capital Greek letters will be used for sets of formulae. As usual, $A \leftrightarrow B$ is defined by $(A \to B) \wedge (B \to A)$. We shall omit parentheses following usual conventions: in particular we assume that $\wedge$ and $\vee$ bind more strongly than $\rightarrow$ and $\leftrightarrow$. The symbols $\forall$, $\exists$, $\Rightarrow$, $\Leftrightarrow$, *and*, *or*, *iff*, *not*, and various set-theoretical symbols will be used in the metalanguage with the usual meaning they have in classical logic. We shall disregard quotation marks in the metalanguage.

Now we introduce the propositional calculus **N** ("**N**" stands for "negation") with the following rules and axiom schemata:

$$\text{MP. } \frac{A \quad A\rightarrow B}{B}, \qquad \text{NR. } \frac{A\rightarrow B}{\neg B\rightarrow\neg A},$$

1. $A\rightarrow(B\rightarrow A)$, 2. $(A\rightarrow(B\rightarrow C))\rightarrow((A\rightarrow B)\rightarrow(A\rightarrow C))$,
3. $(C\rightarrow A)\rightarrow((C\rightarrow B)\rightarrow(C\rightarrow A\wedge B))$, 4. $A\wedge B\rightarrow A$, 5. $A\wedge B\rightarrow B$,
6. $A\rightarrow A\vee B$, 7. $B\rightarrow A\vee B$, 8. $(A\rightarrow C)\rightarrow((B\rightarrow C)\rightarrow(A\vee B\rightarrow C))$,
N1. $\neg A\wedge\neg B\rightarrow\neg(A\vee B)$.

If we omit NR and N1 we obtain an axiomatization of the negationless fragment of the Heyting propositional calculus **H** (for an axiomatization of **H** see ° 7). It is clear that **N**, as well as all subsystems of **H** we shall consider, is a conservative extension of this fragment. By an *extension* of a system **S** we understand any system in which all the theorems of **S** are provable and which is closed under the primitive rules of **S**. We assume throughout this paper that extensions of **N** are in L. It is easy to show that any extension of **N** is closed under the Rule of Replacement

$$\frac{A\leftrightarrow B}{C\leftrightarrow C'}$$

where C' is obtained from C by replacing zero or more occurrences of A in C by B. Next we give the following definition.

Definition 1. If **S** is an extension of **N**, $\Phi \vdash_{\mathbf{S}} A$ iff there is a sequence of formulae $B_1, \ldots, B_n$, $n \geqslant 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ is either a theorem of **S**, or belongs to Φ, or is obtained by MP from formulae preceding it in the sequence.

We shall write $\Phi \nvdash_{\mathbf{S}} A$ instead of *not* $\Phi \vdash_{\mathbf{S}} A$, and $\vdash_{\mathbf{S}} A$, instead of $\varnothing \vdash_{\mathbf{S}} A$. We omit **S** from $\vdash_{\mathbf{S}}$ in contexts where it is clear what system **S** we have in mind. It is easy to prove that the *Deduction Theorem* holds with respect to $\vdash_{\mathbf{S}}$, where **S** is any extension of **N**, i.e. we have

$$\Phi\cup\{A\} \vdash_{\mathbf{S}} B \Rightarrow \Phi \vdash_{\mathbf{S}} A\rightarrow B.$$

(It is essential for this Deduction Theorem that MP is the only rule mentioned in Definition 1.)

A set of formulae Φ is *consistent* (relative to **S**) iff *not* $\forall A \; \Phi \vdash_{\mathbf{S}} A$. Let $Cl(\Phi) =_{\mathrm{df}} \{A \mid \Phi \vdash_{\mathbf{S}} A\}$. It follows immediately that for every Φ, $\Phi \subseteq Cl(\Phi)$ and $Cl(Cl(\Phi)) = Cl(\Phi)$. A set of formulae Φ is *deductively closed* (relative to **S**) iff $Cl(\Phi) \subseteq \Phi$. A set of formulae Φ has the *disjunction property* iff $\forall A, B(A\vee B\in\Phi \Rightarrow A\in\Phi$ *or* $B\in\Phi)$. A system has this property iff the set of its theorems has this property. Using a device like Kleene's slash (cf. [9], pp. 30ft) it is easy to show for **N** and all the subsystems of **H** we shall consider that they have the disjunction property.

§ 2. N models

Models with respect to which we shall show that **N** is sound and complete are defined as follows.

Definition 2. $Fr = \langle X, R_I, R_N\rangle$ is a **N** *frame* iff (i) X is a nonempty set, (ii) $R_I \subseteq X^2$ and R_I is reflexive and transitive, (iii) $R_N \subseteq X^2$, and (iv) $R_I R_N \subseteq R_N R_I^{-1}$ (we use $x, y, z, t, u, v, x_1, \ldots$ as variables ranging over X; expressions of the form $R_1 R_2$, which abbreviates $R_1 \circ R_2$, stand for $\{\langle x, y\rangle \mid \exists z(xR_1 z \textit{ and } zR_2 y)\}$, and those of the form R^{-1} stand for the inverse relation of R).

Definition 3. $M = \langle X, R_I, R_N, V\rangle$ is a **N** *model* iff (i) $\langle X, R_I, R_N\rangle$ is a **N** frame, and (ii) V, called a *valuation*, is a mapping from the set of propositional variables of L to the power set of X such that for every p, $\forall x, y(xR_I y \Rightarrow (x \in V(p) \Rightarrow y \in V(p)))$. Note that R_N in Fr and M can also be empty.

Definition 4. The relation $\langle M, x\rangle \vDash A$ (i.e., *A holds in x in M*), which is usually abbreviated by $x \vDash A$ where there can be no confusion, is defined by

(i) $x \vDash p \Leftrightarrow x \in V(p)$
(ii) $x \vDash B \wedge C \Leftrightarrow x \vDash B$ *and* $x \vDash C$
(iii) $x \vDash B \vee C \Leftrightarrow x \vDash B$ *or* $x \vDash C$
(iv) $x \vDash B \rightarrow C \Leftrightarrow \forall y(xR_I y \Rightarrow (y \vDash B \Rightarrow y \vDash C))$
(v) $x \vDash \neg B \Leftrightarrow \forall y(xR_N y \Rightarrow y \nvDash B)$.

We write $x \nvDash A$ instead of *not* $x \vDash A$.

Definition 5. (i) $M \vDash A$ (i.e., *A holds in M*) iff $\forall x\langle M, x\rangle \vDash A$. (ii) $Fr \vDash A$ (i.e., *A holds in Fr*) iff $\forall M$ (the frame of M is $Fr \Rightarrow M \vDash A$). (iii) A is **S** *valid* iff for every **S** frame Fr, $Fr \vDash A$.

Next it is possible to prove the following lemma by induction on the complexity of A.

Lemma 1 (Intuitionistic Heredity). *In every* **N** *model, for every x and y, and for every A of L,* $xR_I y \Rightarrow (x \vDash A \Rightarrow y \vDash A)$.

This lemma shows that every **N** model is a model for the negationless fragment of **H**. This means that the condition $R_I R_N \subseteq R_N R_I^{-1}$ is sufficient for that to be the case. But this condition is also necessary. Before showing that, we state the following lemma, which we shall have occasions to use also later.

Lemma 2. *Let Q be* $R_1 R_2 \ldots R_n$, $n \geqslant 0$, *where* R_i, $1 \geqslant i \geqslant n$, *is either* R_I, *or* R_N, *or* R_I^{-1}, *or* R_N^{-1}; *and let* $y \in X$, *where* $\langle X, R_I, R_N\rangle$ *is a N frame. Then*

(i) $\forall x(x \vDash p \Leftrightarrow yQR_I x)$ ***or***
(ii) $\forall x(x \vDash p \Leftrightarrow$ ***not*** $xR_I Qy)$ ***or***

(iii) $\forall x(x \vDash p \Leftrightarrow$ ***not*** $yQR_I^{-1}x)$ ***or***
(iv) $\forall x(x \vDash p \Leftrightarrow xR_I^{-1}Qy)$
implies that $\forall x_1, x_2(x_1 R_I x_2 \Rightarrow (x_1 \vDash p \Rightarrow x_2 \vDash p))$.

Now we can show the necessity of the condition $R_I R_N \subseteq R_N R_I^{-1}$ for Intuitionistic Heredity.

Lemma 3. *Let* $\langle X, R_I, R_N\rangle$ *satisfy conditions* (i)—(iii) *of Definition* 2 *and let condition* (iv), *i.e.*, $R_I R_N \subseteq R_N R_I^{-1}$, *be unsatisfied. Then there is a formula A of L and a valuation V such that in* $\langle X, R_I, R_N, V\rangle$ *for some x and y*, $xR_I y$ *and* $x \vDash A$ *and* $y \nvDash A$.

Proof. Since *not* $R_I R_N \subseteq R_N R_I^{-1}$, there are some x, y and z such that

(1) $$xR_I y \text{ and } yR_N z \text{ and } \forall t(xR_N t \Rightarrow \text{not } zR_I t)\,.$$

Let $\forall u(u \vDash p \Leftrightarrow zR_I u)$. By Lemma 2(i) there is a valuation such that this is satisfied From the last conjunct of (1) it follows that with this valuation $x \vDash \neg p$. On the other hand, since $zR_I z$, we have $y \nvDash \neg p$. q.e.d.

In a certain sense we have shown that models with the condition $R_I R_N \subseteq R_N R_I^{-1}$ form the largest class of models with respect to which we can expect to show that **N** is sound and complete. But we also have the following lemmata which indicate that a proper subclass of **N** models might be used as well.

Lemma 4. *In* **N** *models*, $x \vDash \neg A \Leftrightarrow \forall y(xR_N R_I^{-1} y \Rightarrow y \nvDash A)$.

This lemma is easily proved using Intuitionistic Heredity and the reflexivity of R_I. Using the reflexivity and transitivity of R_I we can show the following lemma.

Lemma 5. *In the definition of* **N** *frames we can replace the clause* $R_I R_N \subseteq R_N R_I^{-1}$ *by* $R_I R_N R_I^{-1} \subseteq R_N R_I^{-1}$ *yielding the same class of frames.*

So, roughly speaking, out of **N** models we can make new models by replacing the $R_N R_I^{-1}$ relation by a new relation $R_\neg$ such that in these new models $R_I R_\neg \subseteq R_\neg$, and $R_\neg$ is the R_N relation of the new models, which validate exactly the same formulae as the old ones. Since $R_N R_I^{-1} R_I^{-1} \subseteq R_N R_I^{-1}$, we can further "condense" these models by making $R_\neg R_I^{-1} \subseteq R_\neg$. So, we introduce the following definition.

Definition 6. A **N** frame (model) is *condensed* iff $R_I R_N \subseteq R_N$, and it is *strictly condensed* iff $R_N R_I^{-1} \subseteq R_N$.

It is easy to show that strictly condensed **N** frames form a proper subclass of condensed **N** frames, which form a proper subclass of the class of all **N** frames. It is also easy to show that in condensed **N** frames $R_I R_N = R_N$, whereas in strictly condensed **N** frames $R_I R_N = R_N R_I^{-1} = R_N$. (All the connections between R_I and R_N in strictly condensed **N** frames follow from $R_I R_N R_I^{-1} \subseteq R_N$.)

Another "condensation" of our models would be made by requiring that R_I is not only reflexive and transitive, but a partial ordering. The soundness and completeness results which follow would also hold with such an R_I.

§ 3. Soundness and completeness of N

In this section we shall show that **N** is sound and complete with respect to **N** models, and also with respect to condensed and strictly condensed **N** models. First we introduce the following definition.

Definition 7. A set of formulae Γ is a *theory* iff Γ is deductively closed and has the disjunction property.

In the following lemma (whose analogues are fairly well known; cf. [11], Lemma 2.2) $\vdash$ stands for $\vdash_{\mathbf{S}}$, where **S** is any extension of **N**, and "theory" means "theory with respect to **S**".

Lemma 6. *Let $\Phi \nvdash A$. Then there is a theory Γ such that $\Phi \subseteq \Gamma$ and $\Gamma \nvdash A$ (i.e., $A \notin \Gamma$).*

Proof. Let $Z = \{\psi \mid \Phi \subseteq \psi$ *and* $A \notin \psi$ *and* $Cl(\psi) \subseteq \psi\}$. Since $Cl(\Phi) \in Z$, Z is nonempty, and it is easy to show that it is closed under unions of nonempty chains. Hence, by Zorn's Lemma Z has a maximal element Γ. It is easy to check that Γ is a theory. q.e.d.

On the set of theories we build a canonical model defined as follows.

Definition 8. Let **S** be any extension of **N**, and let

$X^c =_{\mathrm{df}} \{\Gamma \mid \Gamma$ is a theory with respect to $\mathbf{S}\}$

$\Gamma R_I^c \Delta =_{\mathrm{df}} \Gamma \subseteq \Delta$, where $\Gamma, \Delta \in X^c$

$\Gamma R_N^c \Delta =_{\mathrm{df}} \Gamma_{\neg} \cap \Delta = \emptyset$, where $\Gamma_{\neg} =_{\mathrm{df}} \{A \mid \neg A \in \Gamma\}$ and $\Gamma, \Delta \in X^c$.

Then $\langle X^c, R_I^c, R_N^c\rangle$ is the *canonical* **S** *frame*. Let V^c be a mapping from the set of propositional variables of L to the power set of X^c such that $V^c(p) =_{\mathrm{df}} \{\Gamma \mid p \in \Gamma\}$. Then $\langle X^c, R_I^c, R_N^c, V^c\rangle$ is the *canonical* **S** *model.*

This definition of canonical models differs from the usual one in *not* requiring the consistency of theories which make the model. So the set of all formulae, which is a theory, though inconsistent is in the canonical model. In general it will be clear from the context when capital Greek letters range over members of X^c, and we shall not always note specially that the sets in question are theories.

In the following two lemmata **S** stands for any extension of **N**.

Lemma 7. *The canonical* **S** *frame (model) is a strictly condensed* **N** *frame (model).*

Proof. We have that $X^c \neq \emptyset$ since the set of all formulae is a theory. It is trivial to check clauses (ii) and (iii) of Definition 2, and to show that V^c is a valuation. It remains only to check that $\exists\Theta(\Gamma \subseteq \Theta$ *and* $\Theta_{\neg} \cap \Delta = \emptyset) \Rightarrow \Gamma_{\neg} \cap \Delta = \emptyset$, and that $\exists\Theta(\Gamma_{\neg} \cap \Theta = \emptyset$ *and* $\Delta \subseteq \Theta) \Rightarrow \Gamma_{\neg} \cap \Delta = \emptyset$. q.e.d.

Lemma 8. *In the canonical* **S** *model, for every* $\Gamma \in X^c$ *and for every* A, $\Gamma \models A \Leftrightarrow A \in \Gamma$.

Proof. By induction on the complexity of A. We shall consider only the case with $\neg$ of the induction step (the rest is well known; cf. [11], Lemma 2.3). Using the induction hypothesis we have that $\Gamma \models \neg B \Leftrightarrow \forall \Delta(\Gamma_{\neg} \cap \Delta = \emptyset \Rightarrow B \notin \Delta)$. We shall show that $\neg B \in \Gamma \Leftrightarrow \forall \Delta(\Gamma_{\neg} \cap \Delta = \emptyset \Rightarrow B \notin \Delta)$. From left to right this is obvious. For the other direction suppose $\neg B \notin \Gamma$. Then we show that there is a theory Δ such that $\Gamma_{\neg} \cap \Delta = \emptyset$ *and* $B \in \Delta$.

Let $Z = \{\Phi \mid \Gamma_{\neg} \cap \Phi = \emptyset$ *and* $B \in \Phi$ *and* $Cl(\Phi) \subseteq \Phi\}$. First we show that $Cl(\{B\}) \in Z$. The only difficult part of this is to show that $\Gamma_{\neg} \cap Cl(\{B\}) = \emptyset$. Suppose $C \in \Gamma_{\neg}$ *and* $C \in Cl(\{B\})$. Then $\neg C \in \Gamma$ *and* $\{B\} \vdash C$, from which we obtain $\vdash \neg C \rightarrow \neg B$ using the Deduction Theorem and NR. But then since Γ is a theory, $\neg B \in \Gamma$, and this is a contradiction. Hence, Z is nonempty, and it is easy to show that it is closed under unions of nonempty chains. So, by Zorn's Lemma, Z has a maximal element Δ. We shall show that Δ is a theory.

We infer immediately from $\Delta \in Z$ that Δ is deductively closed. To show that it has the disjunction property suppose that for some C and D, $C \vee D \in \Delta$ *and* $C \notin \Delta$ *and* $D \notin \Delta$. Since $\Delta \cup \{C\}$ and $\Delta \cup \{D\}$ are proper supersets of Δ, they cannot be in Z. *A fortiori*, $Cl(\Delta \cup \{C\})$ and $Cl(\Delta \cup \{D\})$ are not in Z. This is possible only if for some C_1 from the first and some D_1 from the second of these last two sets, $\neg C_1 \in \Gamma$ and $\neg D_1 \in \Gamma$. Since Γ is a theory, using N1 we obtain $\neg(C_1 \vee D_1) \in \Gamma$. On the other hand, it is easy to check that $C_1 \vee D_1 \in \Delta$, which contradicts $\Gamma_{\neg} \cap \Delta = \emptyset$. So, Δ has the disjunction property. q.e.d.

Now we can prove the soundness and completeness of **N**.

Theorem 1. $\vdash_{\mathbf{N}} A \Leftrightarrow$ *for every* **N** *frame* Fr, $Fr \models A$
$\Leftrightarrow$ *for every condensed* **N** *frame* Fr, $Fr \models A$
$\Leftrightarrow$ *for every strictly condensed* **N** *frame* Fr, $Fr \models A$.

Proof. The soundness part ($\Rightarrow$) is proved by a straightforward induction on the length of proof of A in **N**. For the completeness part ($\Leftarrow$) suppose $\nvdash_{\mathbf{N}} A$. Then by Lemma 6 it follows that the set of theorems of **N** can be extended to a theory Γ such that $A \notin \Gamma$ (in fact, it is already a theory). Using Lemma 8 we obtain that A does not hold in the canonical **N** model, which according to Lemma 7 means that it doesn't hold in a **N** frame (condensed **N** frame, strictly condensed **N** frame). q.e.d.

§ 4. Soundness and completeness of J

Johansson's, "minimal", propositional calculus **J** will be obtained by extending **N** with the following two schemata

$$A \rightarrow \neg\neg A$$
$$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)\,.$$

It is easy to check that alternatively we could obtain **J** by extending **N** with the following single schema

$$\neg A \leftrightarrow (A \rightarrow \neg(B \rightarrow B))$$

or the following single schema

$$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)\,.$$

Note also that NR and N1 are redundant in the axiomatization of **J**.

In the sequel we shall use $R_\neg$ as an abbreviation for $R_N R_I^{-1}$. This will enable us to translate easily results obtained for **N** models in general into results about strictly condensed **N** models.

Then we show the following lemma.

Lemma 9. *$Fr \models A \rightarrow \neg\neg A \Leftrightarrow R_\neg$ is symmetric.*

Proof. ($\Rightarrow$) Suppose $R_\neg$ is not symmetric. It follows that for some x, y and z, $xR_N z$ *and* $yR_I z$ *and* $\forall t(yR_N t \Rightarrow$ *not* $xR_I t)$. Let $\forall u(u \models p \Leftrightarrow xR_I u)$. By Lemma 2(i) there is a valuation such that this is satisfied. With this valuation it follows from $xR_I x$ that $x \models p$. On the other hand, we obtain $\forall t(yR_N t \Rightarrow t \not\models p)$, and hence $y \models \neg p$. Using Intuitionistic Heredity, it follows that $z \models \neg p$, and since $xR_N z$, we have $x \not\models \neg\neg p$. So, $x \not\models p \rightarrow \neg\neg p$.

($\Leftarrow$) Suppose $v \models A \rightarrow \neg\neg A$. It follows that there is an x such that $vR_I x$ *and* $x \models A$ *and* $x \not\models \neg\neg A$. From the last conjunct we obtain that there is a y such that $xR_N y$ *and* $y \models \neg A$. From $xR_N y$ and the reflexivity of R_I we obtain $xR_\neg y$, and then using the symmetry of $R_\neg$ we obtain $yR_\neg x$, i.e., there is a z such that $yR_N z$ *and* $xR_I z$. From $yR_N z$ and $y \models \neg A$ it follows that $z \not\models A$. But then $xR_I z$ *and* $x \models A$ *and* $z \not\models A$ contradicts Intuitionistic Heredity. q.e.d.

We have proved Lemma 9 in some detail to illustrate the method which can be used to prove a number of similar lemmata about the equivalence of a schema with a condition on **N** frames. We shall state such lemmata in the sequel without proof. For the ($\Rightarrow$) parts we shall need in general Lemma 2 to construct a valuation falsifying the schema in question if the relevant condition doesn't hold. Next we state two such lemmata.

Lemma 10. $Fr \models (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \Leftrightarrow \forall x, y(xR_N y \Rightarrow \exists z(xR_I z$ *and* $yR_I z$ *and* $xR_N z))$.

Lemma 11. (i) $Fr \models \neg A \rightarrow (A \rightarrow \neg(B \rightarrow B)) \Leftrightarrow \forall x, y(\exists z, t(xR_I z$ *and* $yR_I z$ *and* $zR_N t) \Rightarrow \Rightarrow xR_\neg y)$.

(ii) $Fr \models (A \rightarrow \neg(B \rightarrow B)) \rightarrow \neg A \Leftrightarrow \forall x, y(xR_\neg y \Rightarrow \exists z, t(xR_I z$ *and* $yR_I z$ *and* $zR_N t))$.

In accordance with Lemmata 9, 10 and 11, and with what we have stated about the axiomatization of **J**, it is of course possible to show that ($R_\neg$ is symmetric *and* $\forall x, y(xR_N y \Rightarrow \Rightarrow \exists z(xR_I z$ *and* $yR_I z$ *and* $xR_N z)))$ iff $\forall x, y(xR_\neg y \Leftrightarrow \exists z, t(xR_I z$ *and* $yR_I z$ *and* $zR_N t))$.

Next we introduce the following definition.

Definition 9. A **N** frame (model) is a **J** *frame* (*model*) iff $R_{\neg}$ is symmetric and $\forall x, y(xR_N y \Rightarrow \exists z(xR_I z \text{ and } yR_I z \text{ and } xR_N z))$.

We proceed to show that **J** is sound and complete with respect to **J** models. First we prove the following.

Lemma 12. *The canonical* **J** *frame* (*model*) *is a* **J** *frame* (*model*).

Proof. According to Lemma 7, the canonical **J** frame is a strictly condensed **N** frame.

Next we show that in this frame R_N, and hence also $R_{\neg}$, is symmetric. Suppose $\Gamma_{\neg} \cap \Delta = \emptyset$. It follows that $\Delta_{\neg} \cap \Gamma = \emptyset$, since otherwise for some A, $\neg A \in \Delta$ *and* $A \in \Gamma$; using $A \rightarrow \neg\neg A$ we obtain $\neg\neg A \in \Gamma$ *and* $\neg A \in \Delta$, which contradicts $\Gamma_{\neg} \cap \Delta = \emptyset$.

We can also show that $\Gamma_{\neg} \cap \Delta = \Theta \Rightarrow \exists \Theta(\Gamma \subseteq \Theta \text{ and } \Delta \subseteq \Theta \text{ and } \Gamma_{\neg} \cap \Theta = \emptyset)$. Suppose $\Gamma_{\neg} \cap \Delta = \emptyset$ and let $Z = \{\Phi \mid \Gamma \cup \Delta \subseteq \Phi \text{ and } \Gamma_{\neg} \cap \Phi = \emptyset \text{ and } Cl(\Phi) \subseteq \Phi\}$. We can show that $Cl(\Gamma \cup \Delta) \in Z$. The only difficult part of this is to show that $\Gamma_{\neg} \cap Cl(\Gamma \cup \Delta) = \emptyset$. Suppose this is not the case; then for some A, $\neg A \in \Gamma$ and $\Gamma \cup \Delta \vdash A$. Using the deductive closure of Δ (which also implies the nonemptiness of Δ) and the Deduction Theorem, it follows that for some $D \in \Delta$, $\Gamma \vdash D \rightarrow A$. Using $(D \rightarrow A) \rightarrow (\neg A \rightarrow \neg D)$, we obtain $\Gamma \vdash \neg A \rightarrow \neg D$, which with $\neg A \in \Gamma$ and the deductive closure of Γ implies $\neg D \in \Gamma$. But this with $D \in \Delta$ contradicts $\Gamma_{\neg} \cap \Delta = \emptyset$. Hence, Z is nonempty, and it is easy to show that it is closed under unions of nonempty chains. Hence, by Zorn's Lemma, Z has a maximal element Θ. It is easy to check that Θ is a theory (cf. proof of Lemma 8).

Since, as before, V^c is a valuation, this proves the Lemma. q.e.d.

Now, using Lemma 12, and proceeding as for Theorem 1, we obtain the following soundness and completeness theorem.

Theorem 2. $\vdash_J A \Leftrightarrow$ *for every* (*condensed, strictly condensed*) **J** *frame* Fr, $Fr \vDash A$.

§ 5. J models and Q models

Consider the following definition.

Definition 10. $\langle X, R_I, Q\rangle$ is a Q *frame* iff (i) and (ii) are as in Definition 2 and (iii) $Q \subseteq X$ and Q is hereditary, which means $\forall x, y(xR_I y \Rightarrow (x \in Q \Rightarrow y \in Q))$; $\langle X, R_I, Q, V\rangle$ is a Q *model* iff $\langle X, R_I, Q\rangle$ is a Q frame and V is a valuation as in Definition 3.

It is well known that with definitions analogous to Definitions 4 and 5, save that we have

$$x \vDash \neg B \Leftrightarrow \forall y(xR_I y \Rightarrow (y \vDash B \Rightarrow y \in Q))$$

we can show that **J** is sound and complete with respect to Q models (see [11]).

Using Intuitionistic Heredity in Q models and the reflexivity of R_I, it is easy to show the following lemma, which we shall need in the sequel.

Lemma 13. *In Q models*

$$x \models \neg A \Leftrightarrow \forall y (\exists z (x R_I z \text{ and } y R_I z \text{ and } z \notin Q) \Rightarrow y \not\models A).$$

Our purpose now is to show how Q models can be "translated" into strictly condensed **J** models and vice versa. These "translations" are exhibited in the following theorems.

Theorem 3.1. *Let $M_Q = \langle X, R_I, Q, V \rangle$ be a Q model, and let R_N be defined over X by*

(1) $x R_N y \Leftrightarrow \exists z (x R_I z \text{ and } y R_I z \text{ and } z \notin Q)$.

Then $M_N = \langle X, R_I, R_N, V \rangle$ is a strictly condensed **J** *model such that*

(2) $z \in Q \Leftrightarrow \exists x, y (x R_I z \text{ and } y R_I z \text{ and not } x R_N y)$

(3) $\langle M_Q, x \rangle \models A \Leftrightarrow \langle M_N, x \rangle \models A$.

Proof. To show that M_N is a strictly condensed **J** model we first establish that $R_I R_N \subseteq R_N$ and $R_N R_I^{-1} \subseteq R_N$, using the transitivity of R_I. Next, it is clear that $R_\neg$, which equals R_N, is symmetric. Finally, to show $\forall x, y (x R_N y \Rightarrow \exists z (x R_I z \text{ and } y R_I z \text{ and } x R_N z))$ we use the reflexivity of R_I. To show (2) we use the heredity of Q. And to establish (3) it is enough to show, using Lemma 13, that $\langle M_Q, x \rangle \models \neg A \Leftrightarrow \forall y (x R_N y \Rightarrow y \not\models A)$. q.e.d.

Theorem 3.2. *Let $M_N = \langle X, R_I, R_N, V \rangle$ be a strictly condensed* **J** *model, and let $Q \subseteq X$ be defined by (2) of Theorem 3.1. Then $M_Q = \langle X, R_I, Q, V \rangle$ is a Q model such that (1) and (3) of Theorem 3.1 hold.*

Proof. To show that M_Q is a **Q** model it is enough to establish the hereditariness of **Q**, using the transitivity of R_I.

To show (1) from left to right, suppose $x R_N y$. Using the definition of **J** models, it follows that there is a z such that $x R_I z$ *and* $y R_I z$ *and* $x R_N z$. Next suppose $u R_I z$ *and* $v R_I z$. Since R_N is symmetric (remember M_N is strictly condensed), we have $z R_N x$, which in conjunction with $u R_I z$ *and* $v R_I z$, and the condition of Lemma 11(i), implies $u R_N v$. (More precisely, since $z R_N x$, there is a t such that $z R_I t$ *and* $x R_I t$ *and* $z R_N t$. Hence, $u R_I R_N t$, and this implies $u R_N t$. Since $v R_I t$, we have $u R_N R_I^{-1} v$, and so, $u R_N v$.) Hence, *not* $\exists u, v (u R_I z$ *and* $v R_I z$ *and not* $u R_N v)$, i.e., $z \notin Q$.

The other direction of (1) follows trivially. And finally, to establish (3) it is enough to show using (1) that

$$\langle M_N, x \rangle \models \neg A \Leftrightarrow \forall y (\exists z (x R_I z \text{ and } y R_I z \text{ and } z \notin Q) \Rightarrow y \not\models A).$$

Then we apply Lemma 13. q.e.d.

§ 6. N models and some extensions of J

We shall first state some equivalences between schemata and conditions on **N** frames in the style of Lemmata 9, 10 and 11. Analogues of these schemata can be found in [11].

Lemma 14.

(i) $Fr \models A \vee \neg A \Leftrightarrow Fr \models (\neg A \to A) \to A$
$\Leftrightarrow R_{\neg} \subseteq R_I^{-1}$
$\Leftrightarrow R_N \subseteq R_I^{-1}$.

(ii) $Fr \models \neg A \vee \neg\neg A \Leftrightarrow R_{\neg}^{-1} R_{\neg} \subseteq R_{\neg}$ (*i.e.*, $R_{\neg}$ *is euclidean*)
$\Leftrightarrow R_N^{-1} R_N \subseteq R_{\neg}$.

(iii) $Fr \models \neg\neg(\neg(A \to A) \to B) \Leftrightarrow \forall x, y(xR_N y \Rightarrow \exists z(yR_N z \text{ and } \forall t(zR_I t \Rightarrow \exists u\ tR_N u)))$.

The extension of **J** with $A \vee \neg A$ is known as Curry's system **D** (see [5], Chapter 6). This system is sound and complete with respect to (condensed, strictly condensed) **J** models which satisfy $R_N \subseteq R_I^{-1}$. To establish that, it is enough to show that in the canonical **D** frame $R_N^c \subseteq (R_I^c)^{-1}$. Similarly, the extension of J with $\neg A \vee \neg\neg A$ is sound and complete with respect to (condensed, strictly condensed) **J** models in which $R_{\neg}$ is euclidean. Finally, we can show that the extension of **J** with $\neg\neg(\neg(A \to A) \to B)$ is sound and complete with respect to (condensed, strictly condensed) **J** models which satisfy the condition mentioned in Lemma 14(iii).

For this last completeness proof it seems we must use a canonical model made of *consistent* theories only. Such a canonical model could already have been used for the completeness proof of **J**, but it does not seem to be suitable for the completeness proof of **N**. The point is that if we want to use such a canonical model, we must have that every theory contains at least one negated formula in order to show that the Δ constructed in the proof of Lemma 8 is consistent. In the presence of $A \to \neg\neg A$, but also of a weaker schema, like $\neg\neg(A \to A)$, this will be satisfied. Incidentally, we have the following lemma.

Lemma 15. $Fr \models \neg\neg(A \to A) \Leftrightarrow \forall x(\exists y\ yR_N x \Rightarrow \exists z\ xR_N z)$.

The right-hand side of this equivalence amounts to a weak form of *seriality* of R_N (cf. Lemma 16(i)). The extension of **N** with $\neg\neg(A \to A)$ is sound and complete with respect to (condensed, strictly condensed) **N** models which satisfy this weak seriality.

Let us return now to the extension of **J** with $\neg\neg(\neg(A \to A) \to B)$. This system corresponds exactly to the system called **JP**′ in [11] (p. 46), and is the weakest extension of **J** in which $\neg\neg A$ is provable iff A is a classical tautology. The system **JP**′ is sound and complete with respect to the class of Q models which satisfy the condition

$$\forall x, y((xR_I y \text{ and } y \notin Q) \Rightarrow \exists z(yR_I z \text{ and } \forall t(zR_I t \Rightarrow t \notin Q)))$$

(cf. [12]).

We can also obtain completeness proofs with respect to specific classes of **N** models for systems obtained by extending **N**, rather than **J**, with the schemata above.

§ 7. N models and H

The Heyting propositional calculus **H** is obtained by extending **J** with either $\neg(A\to A)\to B$ or $A\wedge\neg A\to B$. For these two schemata we have the following lemma.

Lemma 16. (i) $Fr\models\neg(A\to A)\to B\Leftrightarrow\forall x\exists y\, xR_{\neg}y$

$\Leftrightarrow\forall x\exists y\, xR_N y$ (*i.e.*, $R_{\neg}$, *or* R_N, *is serial*).

(ii) $Fr\models A\wedge\neg A\to B\Leftrightarrow R_{\neg}$ *is reflexive.*

If $R_{\neg}$ is reflexive, it is of course serial, whereas in the presence of

$$\forall x, y(xR_N y\Rightarrow\exists z(xR_I z \text{ and } yR_I z \text{ and } xR_N z))$$

the seriality of $R_{\neg}$ entails its reflexivity.

We shall call **J** frames (models) in which $R_{\neg}$ is reflexive, **H** *frames* (*models*). It is possible to show that **H** is sound and complete with respect to (condensed, strictly condensed) **H** models. (For that we use again a canonical model with consistent theories.)

The relation R_N disappears in a certain sense from **H** models — somewhat analogously to the way Q disappears from Q models adequate for **H**. That is, $R_{\neg}$ becomes definable in terms of R_I, as it is shown by the following lemma, which we state without proof.

Lemma 17. *A* **N** *frame is a* **H** *frame iff* $R_{\neg}=R_I R_I^{-1}$.

This lemma is connected with the fact that in ordinary Kripke models for **H**, of the form $\langle X, R_I, V\rangle$, we have $x\models\neg A\Leftrightarrow\forall y(xR_I R_I^{-1}y\Rightarrow y\not\models A)$, which is easily shown with the help of Intuitionistic Heredity in these models and the reflexivity of R_I (cf. Lemma 13). This also points towards a certain connection between intuitionistic negation and the *Brouwersche* modal logic **B** (based on the classical propositional calculus), for which Kripke frames $\langle X, R_M\rangle$ where R_M is reflexive and symmetric are characteristic. Historically, **B** was connected with intuitionistic negation because $A\to\neg\Diamond\neg\Diamond A$ is provable in **B**, but the converse is not (see [10], p. 58, fn. 37).

Identifying $R_{\neg}$ with $R_I R_I^{-1}$ explains, for example, how the euclideanity of $R_{\neg}$ appears as the condition equivalent with $\neg A\vee\neg\neg A$: in Kripke frames $\langle X, R_I\rangle$ this schema holds iff $R_I^{-1}R_I\subseteq R_I R_I^{-1}$, and this condition is equivalent to $(R_I R_I^{-1})^{-1}R_I R_I^{-1}\subseteq R_I R_I^{-1}$.

§ 8. N models and systems based on classical or intermediate negationless logics

Some well known extensions of **J** or **H** are obtained by adding axioms not involving negation (see [11]). This means that in the corresponding **N** models no new condition involving R_N, but only conditions involving R_I, will be added. These conditions can have repercussions on the R_N relation too. Of course, it is also possible that conditions involving R_N have repercussions on R_I. In this section, with which we shall conclude our paper, we shall make a few comments on these topics.

Curry's system **E** is obtained by extending **J** with $((A \rightarrow B) \rightarrow A) \rightarrow A$ (see [5], Chapter 6). It is easy to conclude that **E** is sound and complete with respect to (condensed, strictly condensed) **J** models in which R_I is an equivalence relation. We can also condense these models with respect to R_I by making R_I an identity relation, as **N** models in general could be condensed by making R_I a partial-ordering relation. Consider **J** frames in which R_I is identity (in these frames $R_I R_N = R_N R_I^{-1} = R_N$ is trivially satisfied). It is easy to infer that in these frames $\forall x, y(xR_N y \Rightarrow x = y)$. Now, the converse of this condition is the reflexivity of R_N, and this is why with the reflexivity of R_N, which comes with $A \wedge \neg A \rightarrow B$, R_N would become the identity relation, and everything would collapse into classical logic.

In general, **N** models with R_I an equivalence, or an identity relation, can serve to study systems which like **E** are obtained by extending the negationless fragment of classical propositional logic with some negation axioms. Similarly, systems related to Dummett's intermediate logic **LC** (see [11]) which are obtained by extending the negationless fragment of **H** plus $(A \rightarrow B) \vee (B \rightarrow A)$ with some negation axioms, could be studied with **N** models where R_I is a linear-ordering relation.

Next we shall mention a condition involving R_N which transforms the R_I relation of a **J** frame into an equivalence relation. This condition is on the right-hand side of the following lemma.

Lemma 18. $Fr \vDash \neg\neg A \rightarrow A \Leftrightarrow \forall x \exists y (xR_N y$ *and* $\forall z(yR_N z \Rightarrow zR_I x))$.

It is well known that either $A \vee \neg A$, or $(\neg A \rightarrow A) \rightarrow A$, or $\neg\neg A \rightarrow A$ can be added to **H** to obtain the classical propositional calculus. But only $\neg\neg A \rightarrow A$ yields this calculus when added to **J** — the other two schemata yield **D**. This also comes out in the condition for $\neg\neg A \rightarrow A$ of Lemma 18, which entails the seriality of R_N. This seriality, or the reflexivity of $R_\neg$ is needed to enable the R_I relation of a **J** frame to collapse into equivalence, or identity.

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